Math Skill #6 Dimensional Analysis

Units can be a very powerful tool for reasoning with numbers, but using them does not always mean simply converting them as we did in Math Skill #5. Sometimes using them means knowing WHEN to convert them, and sometimes it means using units to make sense of what you're doing. The broad set of skills involved in using units to understand measurements, formulae and results is often called "dimensional analysis".

Units can help you figure out what a measurement is really measuring, what a formula really means and whether an answer makes sense. This is true when you are a working out a problem in science AND when you're trying to reason with numbers that you are given in your everyday life.

Most of the numbers we deal with in science are built out of three basic types of units: times, distances and masses. There are of course measurements that include combinations or powers of these base unit types, for example a speed is a distance/time, or a volume is a distance to the third power (cubed), and there are also a few other types of units that we deal with in life more than science (monetary cost, for example). Regardless of the complexity of a problem, it always helps to consider what a number is measuring in a general sense before you do anything with it. Is it a distance, time, speed, volume, cost, count etc? The answer can sometimes tell you what you need to do next!

Consider, for example, that area can be measured in a variety of units, but that they are all squared – square inches, square miles, etc. Where does this come from? Well the unit is telling you something about what area IS on a fundamental level. Because it is measured in units of distance squared, you know that any measurement of area will always include a distance times a distance. For example, if a room is 10ft by 20ft, then we know that we needed to multiply the two measurements together to get an area because we needed to end up with units of ft x ft (or square feet). Therefore we know that we need 10ft x 20ft or 200 square feet of carpet. The type of measurement told us what type of units we needed to end up with and the units we were given then told us what to do with the numbers to get there.

If you had measured one of the dimensions of the room in inches instead of feet, you would have ended up with units of inch-feet when you multiplied them together, and the salesperson at your local Home Depot will have no idea what to do if you tell them that you need 1500inch-feet of carpet. Does this mean that you need to go back and remeasure the room? NO! This is the purpose of a unit conversion! If you convert your inch measurement into feet or your feet to inches, then when you multiply them together you'll get a proper unit of area – either square inches or square feet.

Thus you will often need to recognize when two measurements are measuring the same fundamental quantity – distance, time, mass etc, so that you can match them up. This helps when you are combining them into exponents (feet x feet = feet^2*) and when you are canceling units that appear on the top and bottom of a fraction. No matter* *what its shape, an area will ALWAYS have units of distance squared, thus when you are asked to solve for an area, your final answer should ALWAYS have the proper units for an area – distance squared. If it doesn't, then you should know right away that you need to take another look at how you've approached the problem.*

By applying a little common sense when you are given numbers with units, you can figure out which ones you need, which ones you need to convert and even when there are quantities you can ignore. For example, consider the following problem.

If Suzie runs at a pace of 7 minutes per mile for one hour, how far will she have run? In this case, a careful look at the UNITS in the quantities you're given will tell you how to get at what you are asked for, which is the total distance she has run.

Consider that the 7 minutes per mile has units of time/distance and that 1 hour is a unit of time. Thus you have been given one quantity with units of time/distance and one quantity with units of time, and are asked to solve for a distance. This means that you somehow need to get rid of the time units and end up with distance units alone in your final answer. As you learned with unit conversions, the only way to "get rid" of something is to find it in both the top (numerator) and bottom (denominator) of a fraction so that they cancel one another. This should tell you that you need to divide one of these quantities by the other, but which do you divide by which? This is where careful consideration of units will help you. If you divide the first quantity by the second, you get units of time/distance divided by units of time.

$$\frac{{time}/{distance}}{time}$$

Sidebar 1: Fractions within fractions: "Pie Rules"

In this case, you need to remember your fraction rules OR you can simply use some common sense to work out the fraction rules for yourself. Consider that what you really have above is a fraction divided by a number, and a simple example can tell you how to proceed.

If you start out with half of a pie and you cut it into three pieces, how much of the whole pie does each piece represent? To put it mathematically, what's ½ ÷ 3? In words, what's one third of one-half? Well, logic and experience should tell you that each piece is one-sixth of the whole pie. Thus, when you divide a fraction even further, you should end up with an even smaller fraction. So you combine (multiply) the denominator of the fraction with the number that you are dividing it by. Expressed as a mathematical formula, this is:

$$\frac{A/B}{C} = \frac{A}{B \times C}$$

So what if it is the other way around and you are dividing a number by a fraction? Again, a simple pie example can tell you what to do. If you baked three pies and divided each one into quarters, how many pieces do you have? Mathematically, this is asking the question: what is 3 ÷ ¼ or, in words, how many quarters fit into three wholes? Again, logic and experience should tell you that you will have 12 total slices if you divide three pies into quarters and that when you divide a number by a fraction you get more than you started with. Mathematically, you multiply the denominator of the fraction by the number you are dividing it into to get your final answer. Expressed as a mathematical formula, this is:

$$\frac{A}{B/C} = \frac{A \times C}{B}$$

DO NOT LET THE FORMULAE FOOL YOU INTO THINKING THAT THESE ARE SOMETHING YOU NEED TO MEMORIZE. If the pie exampled above seemed trivial or silly, it's because it was! Don't let fractions throw you just because they're fractions! You don't need to memorize special formulae and mathematical rules in order to figure out math problems in science OR in the real world. You already have the mathematical intuition based on years of experience to answer these questions. The skill is simply in refusing to be cowed by the fact that the problem involves fractions and recall that numbers behave logically! Breathe and think.

Let's return to the problem we started with: $\frac{time/distance}{time}$ *, which we wish to simplify. This is dividing a fraction by a number, so pie rules tell us that this can be simplified to* $\frac{time}{distance \times time}$*. Note that time is on the top and the bottom here, so they cancel and you are left with distance only.*

You might think this means you're done, but if so you didn't look carefully at the fraction, which has distance <u>in the denominator</u>. You want your final answer to have units of distance, not 1/distance (inverse distance), so in fact this was the incorrect choice for how to manipulate the numbers you were given to start with.

Remember that we started with two quantities – one with units of time/distance, and one with units of time and were unsure how to proceed, so we arbitrarily chose to divide the first by the second. WITHOUT PLUGGING IN ANY NUMBERS OR DOING UNIT CONVERSIONS we can tell simply by considering the units that this is not the right thing to do.

What if we had done it the other way around, dividing the time quantity by the time/distance quantity? Well, considering units alone, you'd have

$$\frac{time}{time/_{distance}}$$

Here you have a quantity divided by a fraction, and pie logic will tell you that this simplifies to

$$\frac{time \times distance}{time}$$

Here again, your times cancel, but you have units of distance alone on the top. Hooray! This is what you wanted!

The one remaining caveat is that in order for the times in the top and the bottom to cancel, their units need to be the SAME units of time (minutes, hours, days, etc.), thus you will need to do a unit conversion. The fact that you know in advance that your time units will cancel in the end should reassure you that it really doesn't matter which you convert to which. In this case, you can either convert minutes into hours or vice versa. Your choice is either to convert 7 minutes/mile into hours/mile OR to convert 1 hour into minutes, which in this case is the much simpler choice. In fact, you know right off the bat that there are 60 minutes in one hour.

Recall that we decided based on units alone to divide the time quantity by the time/distance quantity, so to solve this problem you need to divide 60 min by 7 min/mi. We already know that our (matching) time units cancel and we are left with units of distance alone, in this case miles, so all that's left is to divide 60 by 7. This gives you 8.3 miles. If Suzie runs 7 minute miles for 1 hour, she will have covered about 8.3 miles.

Take a moment here to consider that this makes sense. Common sense tells you that if Suzie were a slower runner and were running nice, manageable 10 minute miles for 1 hour, the math would have been much simpler. If she ran 10 minute miles for 60 minutes, then she would have run 6 miles over the course of the hour. She is speedy though, and is running faster than that, thus it makes sense that she covered more distance. Again, don't forget that logic and intuition work with numbers just as they do with words and ideas.

From this point, we could ask a number of simple follow-on questions and answer them with the same basic methodology, such as:

What is Suzie's running pace in mi/hr?
The question is telling you the answer! You know your units need to be mi/hr. You also already know that she covered 8.3 miles in 1 hr, thus her pace is 8.3mi/hr.

How long would it take her to run a marathon (26mi) at this pace?
Here again, you know she covers 8.3mi/hr and that she needs to go a total of 26mi. The problem tells you that you need to end up with units of time alone, so your units of

distance (miles – in this case note that they already match!) need to go away. Do you divide mi/hr by mi or vice versa? Your "pie rules" should tell you! Take a minute and try it. You should get an answer of 3.1hr.

The lesson learned here is to use dimensional analysis to determine which units need to be converted BEFORE you plug anything in.

All we were doing with this problem was solving the formula distance=rate x time, but we didn't even have to remember the formula OR manipulate it with algebra in order to arrive at an answer. ALL WE DID IS CONSIDER UNITS. Also, we didn't do any unit conversions until we knew we needed to. Dimensional analysis will save you a lot of time and effort, particularly as the formulae become more complicated AND it will allow you to make sense of what you are being asked without plugging in any numbers or remembering exact formulae.

Answer the following problems on a separate sheet of paper. Follow the steps outlined below for each problem in order to receive full credit.

- ***(a) Identify what type of base units each quantity has: distance, time, mass, volume, cost, etc. (or some combination).***
- ***(b) Identify what type of units your final answer should be in***
- ***(c) Explain which units you need to get rid of and how to do it***
- ***(d) Explain whether you need any unit conversions to arrive at your answer***
- ***(e) Do the computation and show your work***
- ***(f) Explain IN WORDS why your answer makes sense***

For example, a full solution to the problem we went through in detail is below.

If Suzie runs at a pace of 7 minutes per mile for one hour, how far will she have run?

- *(a) I was given a quantity with units of time/distance and a quantity with units of time.*
- *(b) The solution should have units of distance*
- *(c) I need to get rid of the units of time by dividing so that they appear on the top and the bottom of a fraction. Because I need my final units to be distance (in the numerator), I will divide the time quantity by the time/distance quantity, leaving me with just distance.*
- *(d) I will need to convert minutes to hours or vice versa so that my time units match and cancel.*
- *(e)* $\frac{60min}{7min/_{mi}} = 8.3mi$
- *(f) This makes sense because it is a greater distance than 6 miles, which is the distance that she would have covered if she were running slower 10min miles.*

1. If carpet costs $5/square foot and you need to carpet a room that's 5ft x 10ft, how much will it cost you?
2. The current world population is 7 billion. If the population continues to grow at a rate of 200,000 people per day, then what will the population be in 50 years?
3. Bob has only $20 in his pocket and he is 150 miles from home. His car gets 35 miles per gallon. What is the most that gasoline could cost per gallon for Bob to be able to make it home with his $20?
4. It is approximately 3000miles from coast to coast in the US. If you embark on a cross country road trip where you bike 8hr/day at a speed of 20mi/hr, how long will it take you to cross the country?
5. It costs Suzie $3 per gallon to produce lemonade at her lemonade stand, and she sells 8oz cups for 25cents each. If she makes 3 gallons one Sunday and sells it all, how much profit will she make? (Note: there are 8oz in one cup and 16 cups in a gallon)

Exercise 2: Use this same kind of reasoning for Astronomy problems.
Solve the following problems using whatever method you are most comfortable with.

1. It takes light 8 min to reach the Earth from the Sun, which is 1 AU away. How long does it take light to reach Uranus, which is 20AU from the Sun? Write one sentence explaining why your answer makes sense.
2. The wavelength and frequency of light are related via the speed of light. Write a formula for the speed of light that includes only wavelength and frequency and use units to justify your answer.
3. Someone tells you they heard that the area of the Milky Way Galaxy is 100 thousand light years.
   a. Why do you know, based on units alone, that this can't be true?
   b. Why is "Area" not a good measurement for a 3D object like the Milky Way anyway? Draw a picture to support your answer.
   c. For what kinds of objects is area a good indication of their size? In other words, what kinds of things should you quote areas for instead of lengths or volumes?
4. Someone else tells you that the volume of the observable universe is about 3 trillion trillion million cubic parsecs. What is this as a power of ten and why can you say that the parsec is definitely a unit of distance (even if you didn't remember that to begin with) based on this measurement of volume?
5. The nearest star to our sun is about 3 light years away. If you broadcast a radio signal from Earth and an alien civilization intercepted it and sent another radio signal immediately back, how much older would you be when their reply arrived relative to how old you were when you sent the signal?